Demonstrationes Novae De Resistentia Solidorum

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Mechanics seems to have two parts: one concerned with the power of acting or moving, the other with the power of being acted upon or resisting, or with the firmness of bodies. Few have investigated the latter in much depth. Archimedes, who alone among the ancients worked on the geometry in mechanics, didn't touch upon this part. Since Archimedes almost nothing was achieved in mechanical geometry until Galileo, who, equipped with a discerning judgment and a strong grasp of the geometry of interiors, first put forth the limits of the science, and began to relate the resistance of solids to geometrical laws. And even though he didn't handle this matter or topics surrounding the motion of projectiles exactly, relying on insufficiently certain hypotheses, he nonetheless argued correctly from the foundations he set down.

What he has in mind is the resistance of beams which are fixed to ramparts or walls. In figures 1 and 2, let the beam ABC be fixed perpendicular to the 320
wall or support DE. Let AC equal AB. In figure 1, let there be hung from C the weight F which could precisely tear off a horizontal beam from the wall once erected; and in figure 2, the weight G which could precisely tear off a vertical beam from the horizontal support (I'll call the first of these "breaking off crosswise [*transverse abrumpere*]," the second "tearing off perpendicularly [*directe evellere*]"). According to Galileo, the weight F will be half of the weight G, provided that the solid be fully rigid—i.e. incapable of any bending—and that the weight of the beam itself be set aside or already included in the weight which was hung. Indeed, AB and AC are equal, so the weight F in fig. 1 will incur the same resistance in point B it would if considered perpendicularly, as in fig. 2. Hence let the resistance at point B in either figure be represented by BK, so that the resistance in point H in fig. 2 will be represented by HL, which is equal to BK, because the resistance of all points in fig. 2 is the same. The resistance of the same point H in fig. 1 will be represented by HM, added as an ordinate to the triangle ABK, because it is to the resistance of B as AH is to AB, from the nature of the lever. And as we do the same thing we did with H with any other point between A and B, the square BC will be drawn in order to represent resistance in fig. 2, and the triangle ABK in order to represent resistance in fig. 1, which is half the square. Thus the weight F, if it's assumed to be precisely equal to the resistance in fig. 1, so that adding even the smallest weight would exceed it,

will be half the weight of G (which is precisely equal to the resistance in fig. 2). And the power of breaking off crosswise will be half the power of tearing off perpendicularly (we'll soon find that it's really not half, but a third). Whence many useful conclusions can be deduced.

But Paulus Würzius, who's famous for his first-rank military distinctions and his fairly recent exploits and who also understands these studies very well, has often tried to test these thoughts of Galileo and others of its kind, undertaking many experiments. But his conclusions have hardly yielded any success: I have this from Cl. Blondellus, distinguished in these and other studies, recently appointed mathematics teacher of the most serene *dauphin*, and director of the Architectonic Academy, who developed the same argument and was familiar with Würzius. But Cl. Mariottus of the Regia Academy, clearly excellent on optics and mechanics, also discovered after running experiments that a much lesser weight F was sufficient to break off the beam than what Galileo wanted. 321

The reason for this must be that he considered a perfectly rigid beam, which would be broken off in a single instant once its resistance has become excessive, when all the bodies we have the opportunity to handle yield to some extent before they can be torn off. Observing this, Cl. Mariollus inferred by a clever calculation that the weight F should be about a quarter of the weight G. But since then I've had the chance to consider the matter more deeply and examine it according to the laws of geometry, and I've elicited the true proportions at last and demonstrated among others that the weight F will be a third of the weight G, and therefore that the firmness of bodies resisting rupture is one and a half times less than Galileo wanted.

To see this, one must know before anything else that two bodies adhering to each other are not mutually torn apart as wholes in a single brief instant. This can be seen by the example of a staff which twists before it shatters, and the example of strings, which stretch out before they break; this bend of the staff is a sort of expansion in the arch of its surface. It follows from the nature of sound that nothing is so rigid that it wouldn't still be bent by light pressure, which is a kind of tremor, or a back-and-forth bending of the sounding part of the body—although to the extent that the return to normal is more prompt and less perceptible, and the sound more sharp, to that extent the trembling bits are shorter and more tense, and constitute a more solid body. The long and slender filaments of glass show that it's pliant; the Florentine experiments show how sufficiently dense glass is contracted by the cold. Indeed we learn from the senses alone that parts of plants and animals are in some way like textiles, composed out of various entwined filaments. Minerals and metals too, which were frozen after having been fluids, now have tenacity, and are drawn together in threads, extended with a hammer, and come to adhere in fusion. So let's now reason as though parts of bodies are connected by certain fibers, and understand the beam BC to be bound to the wall or support DE by many braids of fibers in points A, H, B, and the other countless intermediate points. Once the weight F has been hung, the beam will be moved a bit around the fulcrum A, as in fig. 3, and the point B of the beam, separating from the wall from point 1B, on the wall 1B, will arrive at point 2B, away from the wall, carrying with it the fiber with which it is tied to the wall, and will stretch this fiber like a chord, or extend beyond its natural state into the line 1B2B. In the same manner point H will stretch its fiber into 1H2H, which lines, although they're actually imperceptible, have been made visible for the sake of exposition. 322

The line of the 1H2H fiber will resist less pull than the 1B2B fiber, the amount doubling with the distance from A, or out of the doubled extremity from the posited distance. Indeed,

(1) the weight in C which would be needed to stretch the fiber 1H2H as much as the fiber 1B2B would be less than the weight required to stretch the fiber 1B2B that amount. The ratio would be AH:AB; for instance if AH is a third of AB then the weight in C which is able to extend only the 1H2H fiber in such a way that it becomes equal to 1B2B will be a third of the weight extending only the 1B2B fiber. Also,

(2) when both fibers are extended at the same tie by the weight hung at C, surely the fiber 1H2H isn't as tense as the fiber 1B2B, but much less, again in an AH:AB ratio. And so (from the hypothesis (confirmed independently) that extensions are proportional to the forces stretching them) to stretch it we will only need a third of the weight that would have been necessary to stretch 1B2B; i.e. a third of a third of the weight stretching 1B2B, i.e. a ninth thereof.

Thus in general, in the simultaneous tension of all the fibers emerging towards whichever point, the resistances at any given point will be in twice the ratio of the posited distances from the base of the fulcrum, or the center or axis of balance; i.e. the resistance in H will be to the resistance in B as the square AH is to the square AB. Hence if we let the weight F in fig. 3 be the parabolic weight NRSQN, hung freely from C, in which the height NR is equal to the base RS (as AB is equal to AC), and if we add the proportionals as ordinates of the heights

of the square—if PQ is to RS as the square NP is to the square NR—then assuming the base RS represents the resistance in B, the ordinate PQ will represent the resistance in H. If, of course, the heights NP, NR are proportional to the corresponding heights AH, AB. Clearly the entire concave parabolic trilineum NRSQN will represent the resistance of the entire line AB. If, of course, the beam ABC is pushed down by the added weight F crosswise [*transversim*], in the manner of a lever. And the square RNTS circumscribed to this parabolic trilineum would have represented the perpendicular resistance [*resistentia directa*] of the line AB, if, of course, the beam had been pulled directly from the wall, as in fig. 2. Indeed because AB and AC are equal, the crosswise resistance [*resistentia transversa*] of point B will be the same as the perpendicular resistance, obviously represented by RS in fig. 3: if the beam is pulled out perpendicularly (as in fig. 2) the resistance is the same at all points, so the perpendicular 323
resistance at point H will be PV, which is equal to RS. And proceeding in this manner with the rest, the square RT will be completed, which when done will thus be thrice the inscribed parabolic concave trilineum, as NRSQN surely is, and the perpendicular resistance of any straight line (like AB) will thrice the crosswise resistance. QED.

Moreover, however great the length of the beam or the distance of the appended weight from the wall (which thus far we're assumed equal to the height of the beam), it will easily be possible to determine the weight sufficient to break off the beam: so if the weight G can tear off the beam perpendicularly in fig. 4, the weight F will indeed be a third the weight of G (provided AC is equal to AB); but if the weight I is hung from K, and AK is four times AB or AC, the weight I will be a quarter of F, and a twelfth of G. Thus in general the weight tearing off perpendicularly a perpendicular beam will be to the weight breaking off crosswise, in the manner of a lever, as the length of the lever is to the third of the beam's density [*crassities*].

Until now we've considered the beam itself as though it were weightless, but if the weight of the beam should come into account, it will be just as though the weight I of an equal beam had been suspended from K, the center of gravity of this beam. It will also be possible that the beam be broken by its weight in some place—like G in figure 5—between the wall AB and the end of the beam C, when, of course, the gravitation [*gravitatio*] of the FGCF portion, balanced from the point of rest G, has a greater ratio with respect to the resistance in FG than the gravitation of the entire beam BAC from the point of rest AD has with respect to the resistance in AB.

Yet it might be asked what kind of line BFC should be in order that the

resistances be proportional to the corresponding gravitations, and the beam resist equally everywhere: it will be found to be a parabolic one. Indeed the resistance in FG is to the resistance in BA as the parabolic concave trilineum FGHF is to the other such trilineum BAEB, if the base of the trilineum is equal to its height (as is clear from the above) or if it's as the square FG is to the square BA (because the trilineum is such that it's the third of the circumscribed square). But the moment [*momentum*] or gravitation of any FGCF portion balanced from G is to the moment of the entire beam BACB balanced from A as the square FG is to the square BA, as can easily be shown from the nature of the parabola (for the portions CGFC and CABC are like the cubes from CG, CA. And assuming G3 and A2 are a quarter of these (CG and CA), the distances of the centers of gravity of the CGFC and CABC portions from the points of rest or centers of balance G and A will be as well, and the moments of the 324
said portions are as follows from the portions with respect to their distances, or in the combined sum of the portions or cubes from CG and CA and of the distances, which are as CG and CA themselves.) Hence the resistances are proportional to the moments or forces [*vires*], the proportion of some moment to its resistance is everywhere the same, and indeed the firmness [*firmitas*] by which the beam everywhere resists its own weight will also be equal. Therefore let the beam run however long you can imagine: if it doesn't break under its weight near the wall, it won't break anywhere else.

Furthermore if the beam CABC is a prismatic parabolic third of the CDBA solid [*plena*], so that a third of the weight has been removed from that part and the distance of the center of gravity has been drawn back from AG to its half A2, the parabolic beam will be six times firmer than the plane. But if the force of the beam is taken to have negligible weight, as of water or wind, or something else distributed equally across the whole length of the beam, as if in fig. 6 the beam ABD running out of the wall should support the strain of soil thrown upon it, or of grain or some other material, it will be possible for it to be triangular, and the line AD straight, and the beam will resist everywhere equally to the weight placed upon it, since if it doesn't break in the wall it's impossible for it to break anywhere else: indeed from the known laws of mechanics, the moment of the weight pressing upon GD is to the moment of the weight pressing upon BG as the square GD is to the square BD, or as the square GF is to the square BA; i.e. as the resistance at GF is to the resistance at BA: so whether the placed weight is under consideration or the shape of the beam, I'm just as able to provide a shape resisting equally.

Thus far we've only considered a beam the surface of which adheres to a wall or support and is everywhere of equal height, so that it sufficed to assume BA straight. But because the shared surface of the beam and wall can vary, we offer a general rule to determine its resistance geometrically, a special case of which will lead anyone who has the time to draw out its consequences to discover many highly elegant theorems.

In this way, let ABHC be a beam, as in fig. 7, which intersects the support DE in a plane ABH of any given shape. Let that plane be dragged horizontally and let there be drawn another plane equal to it and similar in the horizontal plane, and let AGH be similarly placed. From the point G, construct the lowest of the horizontals, most distant from AH (the point B corresponds to this), and GF (equal to BF) perpendicular to AH. Let the cylindrical body whose base or section of whatever sort is parallel to the horizontal plane be similar 325
and equal to AGH, and GI be perpendicular to its height and equal to FG or BF. One can call this body a cylinder. Construct through an indefinite tangent KIL parallel to AH. Finally let a plane cross AH and KL, which will make a 45 degree angle with the horizontal plane and will cut the cylindrical body in two parts, of which the one where GI falls, which in the figure is the secant above the plane, is called an ungula by geometers. I claim that this ungula, cut back from the cylinder and serving as a lever whose fulcrum lies in AH, is equal to or represents the resistance of the beam ABHC to being broken off crosswise in AHB, provided the weight of the cylinder itself is enough to tear off that same beam perpendicularly from the wall.

Though it's helpful to consider the ungula in the manner of a lever, so that we may consider completely the weight representing resistance let the ungula be hung from the point M, or from FM, the distance from the wall to the center of gravity of the ungula; it will thus be precisely equal to the crosswise resistance, if the whole cylinder is equal to the perpendicular. Therefore when we ask whether and where some solid should break, it won't be difficult to give a geometrical assessment. For it will come out the same whether it's the strongest point or not: wherever the moment of the ungula, or what's made of the ungula by drawing it out until its center of gravity, will have the least ratio to the power attempting to break it there away from the vertical plane in which lies the axis of balance of everything: therefore by these few considerations this whole matter is brought back to pure geometry, which is especially lacking in physics and mechanics.

Supplement: if someone asked for some conoeide [cf. *Errata*] equal to this resistance, a parabolic tube will be satisfactory. In fig. 8, let AEC be a parabolic line whose vertex A is tangent to the vertex AB, and rotate the parabolic line as you would around an axis. This will give you the AECGDFA tube. When the other AEHFA portion of the tube is placed, since the resistances of the bases or circles CGD, EHF are as the cubes of the diameters CD, EF it will be found that the moments of the portions AECGDFA and AEHFA are also like the cubes CD, EF, by the nature of the parabola.